A survey on borderenergetic graphs

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Outline

1. Preliminary
2. Computer search and construction
3. Lower bounds on the size
4. Complements of borderenergetic graphs
5. Laplatian borderenergetic graphs
6. Q-borderenergetic graphs

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Preliminary

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- Ivan Gutman proposed the concept of graph energy in 1977, see [I. Gutman, Acylclic systems with extremal Hückel π-electron energy, Theor. Chim. Acta. 45 (1977) 79–87]

  The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of the eigenvalues of G, i.e., $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$.

Borderenergetic graphs

- In 2015, Gutman et al. launched the study of graphs with energy equal to that of the complete graph.

Borderenergetic graphs

- In 2015, Gutman et al. launched the study of graphs with energy equal to that of the complete graph.
- An n-vertex graph G is called a borderenergetic graph if $\mathcal{E}(G) = \mathcal{E}(K_n) = 2(n-1)$, where K_n is the complete graph of order n. A borderenergetic graph is said to be noncomplete if it is different from the complete graph K_n.

Borderenergetic graphs

[1]S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, Borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 321–332.

[2]X. Li, M. Wei, S. Gong, A computer search for the borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 333–342.

[3]Z. Shao, F. Deng, Correcting the number of borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 263–266.

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Borderenergetic graphs

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Computer search

S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, Borderenergetic graphs, MATCH Commun. Math. Comput. Chem. 74 (2015) 321–332 established

- there are no noncomplete borderenergetic graphs of order less than 6;

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- there are 1, 6 and 17 noncomplete borderenergetic graphs of orders 7, 8 and 9, respectively;

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- there are no noncomplete borderenergetic graphs of order less than 6;
- there are 1, 6 and 17 noncomplete borderenergetic graphs of orders 7, 8 and 9, respectively;
- all these graphs are given in the same paper.

Computer search

X. Li, M. Wei, S. Gong, A computer search for the borderenergetic graphs of order 10, MATCH Commun. Math. Comput. Chem. 74 (2015), 333–342
and
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- computer-aided searching for graphs on $n = 10$ and $n = 11$ vertices and obtain

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- there exist exactly 49 and 158 non-isomorphic borderenergetic graphs of orders 10 and 11, respectively.

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performed

- computer-aided searching for graphs on $n = 10$ and $n = 11$ vertices and obtain
- there exist exactly 49 and 158 non-isomorphic borderenergetic graphs of orders 10 and 11, respectively.
- So, all the borderenergetic graphs of order $n \leq 11$ have been completely found.

Construction

- The tensor product of two graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.

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Theorem 2.1 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

Let G be a borderenergetic graph. Suppose that G is obtained from the tensor product of two integral graphs G_1 and G_2. Then both $|G_1|$ and $|G_2|$ are odd numbers.

Construction

- Let G be a regular graph that is neither complete nor empty. Then G is said to be a strongly regular graph with parameters (n, k, a, c) if it is k-regular with order n, every pair of adjacent vertices has a common neighbors, and every pair of distinct non-adjacent vertices has c common neighbors.

Construction

- Let G be a regular graph that is neither complete nor empty. Then G is said to be a strongly regular graph with parameters (n, k, a, c) if it is k-regular with order n, every pair of adjacent vertices has a common neighbors, and every pair of distinct non-adjacent vertices has c common neighbors.
- The eigenvalues of a strongly regular graph with parameters (n, k, a, c) are k with multiplicity 1, $[(a-c)+\sqrt{\Delta}]/2$ with multiplicity m_θ and $[(a-c)-\sqrt{\Delta}]/2$ with multiplicity m_τ where $\Delta = (a-c)^2+4(k-c)$ and m_θ, m_τ satisfy the equations $m_\theta + m_\tau = n-1, m_\theta\theta + m_\tau\tau = -k$. A strongly regular graph with $m_\theta = m_\tau$ is called a conference graph.

Construction

Theorem 2.2 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

Let G be a conference graph. If G is integral and noncomplete borderenergetic, then G has parameters $(9, 4, 1, 2)$.

Construction

Considering the line graph, The following fact has been found

Theorem 2.3 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

The line graph of the Petersen graph is a connected noncomplete borderenergetic graph.

Using the union of graphs, Gong et al. showed the following result.

Theorem 2.4 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

For every integer n with $n \geq 13$, there exists a noncomplete borderenergetic graph of order n.

Construction

Considering a regular graph and using the spectrum of its complement, a lot of resluts have been obtained

Theorem 2.5 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

Let $p, q,$ and r be non-negative integers, and let $p + q = 2$. Then $\overline{pC_4 \cup qC_6 \cup rC_3}$ is borderenergetic.

Theorem 2.6 (S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, MATCH 2015)

For every integer n with $n \geq 7$, there exists a noncomplete borderenergetic graph of order n.

Construction

Theorem 2.7 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let k be an even integer. Let $G = pG_1 \cup qK_{k+1}$ be a disconnected k-regular graph consisting of p copies of G_1 and q copies of K_{k+1}, where G_1 be a connected k-regular integral graph with $k+2$ vertices, having t_1 non-negative eigenvalues, and satisfying $\mathcal{E}(G_1) = 2k + 4 - 2t_1 + \frac{2k}{p}\, , p|2k\, ,\ p \geq 1\, , q \geq 1$. Then $\overline{G}$ is a connected non-complete borderenergetic graph.

Theorem 2.8 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let G be a k-regular integral graph of order n with t non-negative eigenvalues. If $\mathcal{E}(G) = 2(n - t + k)$, then $\mathcal{E}(\overline{G}) = 2(n - 1)$.

Construction

Theorem 2.9 (B. Deng, X. Li, I. Gutman, LAA 2016)

For integer n *(*$n > 12$*) satisfying* $5|(n-12)$*, there exists a connected non-complete* $(n-5)$*-regular borderenergetic graphs of order* n*.*

Theorem 2.10 (B. Deng, X. Li, I. Gutman, LAA 2016)

For integer n *(*$n > 16$*) satisfying* $7|(n-16)$*, there exists a connected non-complete* $(n-7)$*-regular bordernergetic graphs of order* n*.*

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Lower bounds on the size

- For an integer $r \geq 0$, the *r-degree* $d_r(v_i)$ of a vertex $v_i \in G$ is defined as the number of walks of length r starting at v_i. Clearly, one has $d_0(v_i) = 1, d_1(v_i) = d_i$ and $d_{r+1}(v_i) = \sum_{w \in N(v_i)} d_r(w)$, where $N(v_i)$ is the set of all neighbors of the vertex v_i. In addition, we denote $d_2(v_i)$ and $d_3(v_i)$ by t_i and σ_i for $v_i \in V(G)$, respectively.

Lower bounds on the size

Theorem 3.1 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let G be a borderenergetic graph. Then

$$m \geq \left\lceil \frac{1}{2}\Delta + \frac{1}{2(n-1)}\left(2(n-1) - \sqrt{\Delta}\right)^2 \right\rceil.$$

where $\Delta = \frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}$. *If G is $(n-3)$-regular, then this bound is asymptotically tight.*

Lower bounds on the size

Theorem 3.2 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let G be a borderenergetic graph of order n. Then

$$m \geq \left\lceil \frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right\rceil .$$

If G is $(n-3)$-regular, then this bound is asymptotically tight.

Lower bounds on the size

Theorem 3.3 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let G be a borderenergetic graph. Then

$$m \geq \left\lceil \frac{1}{2} \sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2} \right)^2 \right\rceil .$$

If G is $(n-3)$-regular, then this bound is asymptotically tight.

Lower bounds on the size

Theorem 3.4 (B. Deng, X. Li, I. Gutman, LAA 2016)

Let G be a borderenergetic graph. Then

$$m \geq \left\lceil \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2} \right)^2 \right\rceil .$$

If G is $(n-3)$-regular, then this bound is asymptotically tight.

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Theorem 4.1 (B. Zhou, I. Gutman, Bull. Acad. Serbe Sci. Arts 2007)

Let G be a graph with n vertices. Then

$$\varepsilon(G) + \varepsilon(\overline{G}) < \sqrt{2}n + (n-1)\sqrt{n-1}.$$

From this we can immediately get

Corollary 4.2

If G is a borderenergetic graph with n vertices, then

$$\varepsilon(\overline{G}) < \sqrt{2}n + (n-1)(\sqrt{n-1} - 2).$$

Let ω and $\overline{\omega}$ be the clique numbers of G and $\overline{G}$, respectively.

Theorem 4.3 (B. Zhou, I. Gutman, Bull. Acad. Serbe Sci. Arts 2007)

Let G be a graph with n vertices. Then

$$\varepsilon(G) + \varepsilon(\overline{G}) < \sqrt{(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}})n(n-1)} + (n-1)\sqrt{n-1}.$$

With the same reason, we immediately obtain

Corollary 4.4

If G is a borderenergetic graph with n vertices, then

$$\varepsilon(\overline{G}) < \sqrt{(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}})n(n-1)} + (n-1)(\sqrt{n-1} - 2).$$

Theorem 4.5 (B. Deng, X. Li, MATCH 2020)

If G is a borderenergetic regular graph with n vertices, then

$$\varepsilon(\overline{G}) < (n-1)(\sqrt{n+1}-1).$$

Theorem 4.6 (B. Deng, X. Li, MATCH 2020)

If G is a k-regular borderenergetic graph with n vertices, then

$$\varepsilon(\overline{G}) \leq 4(n-1) - 2k.$$

Theorem 4.7 (B. Deng, X. Li, MATCH 2020)

Let G be a graph of order n. Then

$$\varepsilon(G)+\varepsilon(\overline{G}) < \frac{4}{3}n-1 + \sqrt{2n(n-1)^2-(n-1)[n-1+\frac{\sqrt{2}s^2(G)}{n^2\sqrt{n(n-1)}}]^2}.$$

where $s(G)=\sum_{1\leq i\leq n}|d_i-\frac{2m}{n}|$.

Observing the following two types of strongly regular graphs, it is easy to check that the upper bound is asymptotically tight.

Type 1. If G is a strongly regular graph with parameters $(n, (n+\sqrt{n})/2, (n+2\sqrt{n})/4, (n+2\sqrt{n})/4)$, then

$$\varepsilon(G) + \varepsilon(\overline{G}) = (n-1)(\sqrt{n}+1) - 1.$$

Type 2. For a Paley graph H, which is a strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$,

$$\varepsilon(H) + \varepsilon(\overline{H}) = (n-1)(\sqrt{n}+1).$$

If G is borderenergetic, by above Theorem 2.7, we get

Theorem 4.8 (B. Deng, X. Li, MATCH 2020)

Let G be a borderenergetic graph. Then

$$\varepsilon(\overline{G}) \quad < \quad \sqrt{2n(n-1)^2 - (n-1)[n-1+\frac{\sqrt{2}s^2(G)}{n^2\sqrt{n(n-1)}}]^2} - \frac{2n}{3} + 1.$$

From the above inequality, as n is large enough, it yields that

$$\varepsilon(\overline{G}) \quad < \quad O(n^{3/2}).$$

In fact, above inequality can be better improved as follows.

Theorem 4.9 (B. Deng, X. Li, MATCH 2020)

Let G be a borderenergetic graph of order n. Then

$$\varepsilon(\overline{G}) = O(1),$$

as n is large enough.

What is more important, by above Theorem 4.9, we see that, in the case of borderenergetic graphs, the upper bound behaves much better for the Nordhaus-Gaddum-type result. That is

Theorem 4.10 (B. Deng, X. Li, MATCH 2020)

Let G be a borderenergetic graph with n vertices. Then

$$\varepsilon(G) + \varepsilon(\overline{G}) < O(n),$$

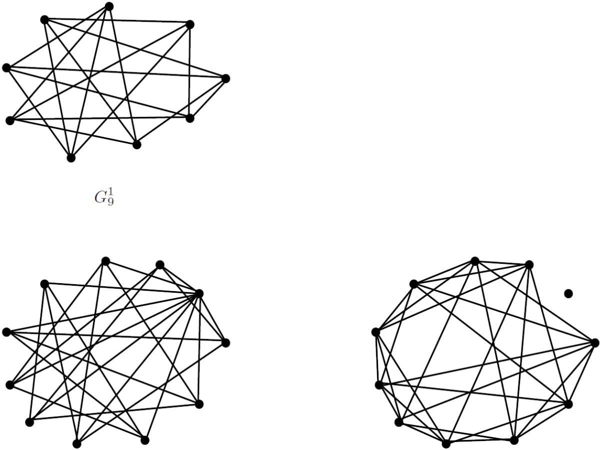
as n is large enough.

An interesting question is whether there is a graph G such that both G and its complement $\overline{G}$ are borderenergetic ? In fact, if it is yes, then one will have

$$\varepsilon(G) + \varepsilon(\overline{G}) = 4(n-1),$$

which is in linear of n.

In fact, borderenergetic graphs of order n with $1 \leq n \leq 11$ were found by using computers in [1-3], and there is no borderenergetic graph of order less than 7. One can check that among them, only three graphs have the property that both the graph and its complement are borderenergetic. The adjacency spectra of the three graphs are given as follows:



G_9^1 G_{11}^2 G_{11}^3

Figure 1. Three borderenergetic graphs: G_9^1, G_{11}^2 and G_{11}^3.

$$Sp_A(G_9^1) = \{4, 1, 1, 1, 1, -2, -2, -2, -2\};$$
$$Sp_A(G_{11}^2) = \{5, 1, 1, 1, 1, 1, -2, -2, -2, -2, -2\};$$
$$Sp_A(G_{11}^3) = \{6, 1, 1, 1, 1, 0, -2, -2, -2, -2, -2\}.$$

One can see that $\overline{G_9^1} \cong G_9^1$ and $\overline{G_{11}^2} \cong G_{11}^3$, and especially, the graph G_9^1 is self-complementary. So, we are left to deal with the case of $n \geq 12$ only.

Theorem 4.11 (B. Deng, X. Li, MATCH 2020)

Except for the three graphs G_9^1, G_{11}^2 and G_{11}^3, for any graph G at most one of G and its complement $\overline{G}$ can be a borderenergetic graph.

From the above result, one can immediately derive the following corollaries.

Corollary 4.12 (B. Deng, X. Li, MATCH 2020)

There is a unique self-complementary borderenergetic graph, which is the graph G_9^1 on 9 vertices.

Corollary 4.13 (B. Deng, X. Li, MATCH 2020)

There is a unique regular borderenergetic graph for which both the graph and its complement are borderenergetic, which is the graph G_9^1 on 9 vertices.

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Theorem 5.1 (X. Lv, B. Deng, X. Li, MATCH 2021)

For any connected noncomplete graph G, except for the four graphs G_5^1, G_6^2, G_6^3 and G_9^4, at most one of G and its complement $\overline{G}$ can be L-borderenergetic.

Proof

Proof. For the cases of $4 \leq n \leq 11$, all the L-borderenergetic graphs have been found. Let G be an L-borderenergetic graph with order $n \geq 12$. By contradiction, suppose $\overline{G}$ is also an L-borderenergetic graph, i.e., $LE(\overline{G}) = 2(n-1)$. By $LE(\overline{G}) \geq 2\sqrt{M(\overline{G})}$, we have $2(n-1) \geq 2\sqrt{M(\overline{G})}$ and

$$(n-1)^2 \geq M(\overline{G}). \tag{1}$$

Let $\overline{m}$ be the size of $\overline{G}$. The degree of a vertex v_i in $\overline{G}$ is denoted by $\overline{d_i}$. By investigating the quality $M(\overline{G})$, we obtain

$$
\begin{aligned}
M(\overline{G}) &= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}(\overline{d_i} - \frac{2\overline{m}}{n})^2 \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}(\overline{d_i}^2 - \frac{4\overline{m}}{n}\overline{d_i} + \frac{4\overline{m}^2}{n^2}) \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2 - \frac{2\overline{m}}{n}\sum_{i=1}^{n}\overline{d_i} + \frac{2\overline{m}^2}{n} \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}(n-1-d_i)^2 - \frac{2\overline{m}}{n}\sum_{i=1}^{n}(n-1-d_i) + \frac{2\overline{m}^2}{n}
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left[(n-1)^2 - 2(n-1)d_i + d_i^2\right] - 2\overline{m}(n-1) + \frac{2\overline{m}}{n}\sum_{i=1}^{n} d_i \\
&= \frac{1}{2}n(n-1) - m + \frac{1}{2}n(n-1)^2 - 2m(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2 - 2(n-1) \\
&\quad + \frac{4m}{n}\left[\frac{1}{2}n(n-1) - m\right] + \frac{2}{n}\left[\frac{1}{2}n(n-1) - m\right]^2 \\
&= -\frac{2}{n}m^2 - m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + \frac{1}{2}n(n-1).
\end{aligned}
$$

Assume $\sum_{i=1}^{n} d_i^2 = x$. Then

$$M(\overline{G}) = -\frac{2}{n}m^2 - m + \frac{1}{2}x + \frac{1}{2}n(n-1). \tag{2}$$

Combining (1) with (2), it arrives at

$$\begin{aligned}(n-1)^2 - M(\overline{G}) &= n^2 - 2n + 1 + \frac{2}{n}m^2 + m - \frac{1}{2}x - \frac{1}{2}n^2 + \frac{1}{2}n \\ &= \frac{2}{n}m^2 + m - \frac{1}{2}x + \frac{1}{2}n^2 - \frac{3}{2}n + 1 \geq 0. \end{aligned} \tag{3}$$

Since $\sum_{i=1}^{n} d_i = 2m$, we get

$$\begin{aligned}(\sum_{i=1}^{n} d_i)^2 &= 4m^2 = \sum_{i=1}^{n} d_i^2 + 2\sum_{i\neq j} d_i d_j \leq x + n(n-1)^3 \\ &= x + n^4 - 3n^3 + 3n^2 - n.\end{aligned}$$

Thus,

$$m \leq \sqrt{\frac{1}{4}x + \frac{n^4 - 3n^3 + 3n^2 - n}{4}}.$$

By (3), we obtain

$$\frac{2}{n}(\frac{1}{4}x + \frac{n^4 - 3n^3 + 3n^2 - n}{4}) + \sqrt{\frac{1}{4}x + \frac{n^4 - 3n^3 + 3n^2 - n}{4}}$$

$$-\frac{1}{2}x + \frac{1}{2}n^2 - \frac{3}{2}n + 1 \geq 0.$$

So,

$$\frac{(n-1)^2}{4n^2}x^2 - \frac{2n^4 - 6n^3 + 4n^2 + 3n - 2}{4n}x + \frac{n^6 - 4n^5 + 3n^4 + 5n^3 - 7n^2 + n + 1}{4} \leq 0.$$

The left expression of the above inequality can be seen as a function with variable x, i.e., $f(x)$. Then the above inequality can be written as

$$f(x) \leq 0.$$

Obviously, we can see that the discriminant Δ of the quadratic equation $f(x) = 0$ satisfies

$$\Delta = \frac{4n^6 - 16n^5 + 28n^4 - 32n^3 + 25n^2 - 8n}{16n^2} > 0,$$

which implies that there are solutions for the inequality $f(x) \leq 0$.

Let $x_1 < x_2$ be two roots of the equation $f(x) = 0$, possessing that $x_1 \leq x \leq x_2$. It is not hard to find that

$$x_1 = \frac{2n^5 - 6n^4 + 4n^3 + 3n^2 - 2n}{2(n-1)^2} - \frac{2n^2}{(n-1)^2}\sqrt{\Delta},$$

$$x_2 = \frac{2n^5 - 6n^4 + 4n^3 + 3n^2 - 2n}{2(n-1)^2} + \frac{2n^2}{(n-1)^2}\sqrt{\Delta}.$$

If $\Delta(G) \leq n-2$, then $x \leq n(n-2)^2$. As $n \geq 12$, we have

$$\begin{aligned} x_1 - n(n-2)^2 &= \frac{2n^5 - 6n^4 + 4n^3 + 3n^2 - 2n}{2(n-1)^2} \\ &\quad - \frac{2n^2}{(n-1)^2}\sqrt{\Delta} - n(n-2)^2 \\ &= \frac{n(-10 + 27n - 22n^2 + 6n^3 - 4n\sqrt{\Delta})}{2(n-1)^2} > 0, \end{aligned}$$

which is a contradiction with $x_1 \leq x \leq n(n-2)^2$.

Next, we consider the case of $\Delta(G) = n - 1$. Note that the number of vertices with degree equal to $n-1$ is at most $n-2$. Thus, we have $x \leq (n-2)(n-1)^2 + 2(n-2)^2$. Combining $(n-2)(n-1)^2 + 2(n-2)^2 < x_2$, we get

$$x \leq (n-2)(n-1)^2 + 2(n-2)^2 < x_2,$$

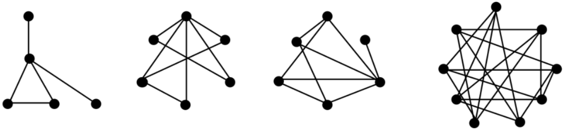
and

$$f((n-2)(n-1)^2 + 2(n-2)^2) \leq 0. \tag{4}$$

But for $n \geq 12$, it holds that

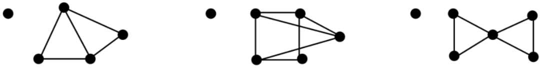
$$f((n-2)(n-1)^2 + 2(n-2)^2) = \frac{2n^5 - 8n^4 + 35n^2 - 48n + 18}{2n^2} > 0,$$

which creates a contradiction with (4). $\square$



G_5^1 G_6^2 G_6^3 G_9^4

Figure2. The L-borderenergetic graphs: G_5^1, G_6^2, G_6^3 and G_9^4.



$\overline{G}_5^1$ $\overline{G}_6^2$ $\overline{G}_6^3$

Figure3. The complements of graphs G_5^1, G_6^2 and G_6^3.

The Laplacian spectra of the four graphs and their complements are given, respectively, as follows.

$$
\begin{aligned}
LS_p(G_5^1) &= \{5,3,1,1,0\};\\
LS_p(G_6^2) &= \{6,4,3,2,1,0\};\\
LS_p(G_6^3) &= \{6,5,3,3,1,0\};\\
LS_p(G_9^4) &= \{6,6,6,5,5,3,3,2,0\};\\
LS_p(\overline{G}_5^1) &= \{4,4,2,0,0\};\\
LS_p(\overline{G}_6^2) &= \{5,4,3,2,0,0\};\\
LS_p(\overline{G}_6^3) &= \{5,3,3,1,0,0\};\\
LS_p(\overline{G}_9^4) &= \{6,6,6,5,5,3,3,2,0\}.
\end{aligned}
$$

The auxiliary quantity $M(G)$ of G is defined as

$$M(G) = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2.$$

Similarly,

$$M(\overline{G}) = \overline{m} + \frac{1}{2}\sum_{i=1}^{n}(\overline{d_i} - \frac{2\overline{m}}{n})^2.$$

These two kinds of inequalities below can be shown as follows.

The Koolen-Moulton type of inequality on the Laplacian energy is

$$\mathcal{LE}(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2M - (\frac{2m}{n})^2]}.$$

The McClelland type of inequality on the Laplacian energy is

$$\mathcal{LE}(G) \leq \sqrt{2nM}.$$

Denote the maximum degrees of G and $\overline{G}$ by Δ and $\overline{\Delta}$, respectively. Suppose $\Delta_0 = max\{\Delta, \overline{\Delta}\}$. We get a Nordhaus-Gaddum-Type bound for the Laplacian energy as follows.

Theorem 5.2

Let G be a graph of order n. Then

$$\mathcal{LE}(G) + \mathcal{LE}(\overline{G}) \quad < \quad n - 1 + 2(n-1)\sqrt{\frac{n}{2}(\Delta_0 + 1) - 1 - \frac{1}{n}}.$$

By Theorem 4.1, it easy to see that

Corollary 5.3

Let G be an L-borderenergetic graph of order n. Then

$$\mathcal{LE}(\overline{G}) \quad < \quad 2(n-1)\sqrt{\frac{n}{2}(\Delta_0 + 1) - 1 - \frac{1}{n}} - n + 1.$$

On the other hand, from the McClelland type of inequality on the Laplacian energy, a better bound on $\mathcal{LE}(G)+\mathcal{LE}(\overline{G})$ is presented.

Theorem 5.4

Let G be a graph of order n. Then

$$\mathcal{LE}(G)+\mathcal{LE}(\overline{G}) \quad < \quad n\sqrt{2(n-1)(\Delta_0+1-\frac{2}{n})}.$$

From Theorem 4.3, it directly yields that

Corollary 5.5

Let G be an L-borderenergetic graph with n vertices. Then

$$\mathcal{LE}(\overline{G}) \quad \leq \quad n\sqrt{2(n-1)(\Delta_0+1-\frac{2}{n})}-2(n-1).$$

Outline

1 Preliminary

2 Computer search and construction

3 Lower bounds on the size

4 Complements of borderenergetic graphs

5 Laplatian borderenergetic graphs

6 Q-borderenergetic graphs

Recently, Tao and Hou[13] extend this concept to the signless Laplacian energy of a graph. If a graph has the same signless Laplacian energy as the complete graph K_n, i.e.,

$$QE(G) = QE(K_n) = 2(n-1),$$

then it is called *Q-borderenergetic.*

[13]Q. Tao, Y. Hou, Q-borderenergetic graphs, *AKCE International Journal of Graphs and Combinatorics*, doi.org/10.1016/j.akcej.2018.03.001, 2018.

The signless Laplacian spectrum of the complement of any k-regular graph G with order n is given in the following lemma. Denote the signless Laplacian matrix of $\overline{G}$ by $\overline{Q}$.

Lemma 6.1 (B. Deng, C. Chang, et al. 2020+)

Let G be a k-regular connected graph of order n. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the eigenvalues of $Q(G)$, then the eigenvalues of $Q(\overline{G})$ are as follows:

$$2(n-1)-\mu_1 = 2(n-1-k) \geq n-2-\mu_n \geq n-2-\mu_{n-1} \geq \cdots \geq n-2-\mu_2.$$

Using Lemma 2.1, we obtain the signless Laplacian spectrum of the join of two special graphs in the following theorem.

Theorem 6.2 (B. Deng, C. Chang, et al. 2020+)

Let G_1 be a k-regular graph on n vertices and G_2 be an empty graph on $n-k$ vertices. If $2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the signless Laplacian eigenvalues of G_1, then the signless Laplacian eigenvalues of $G_1 \nabla G_2$ are

$$n-k+\mu_2,\ n-k+\mu_3, \ldots,\ n-k+\mu_n,\ n^{(n-k-1)},\ k,\ 2n.$$

Using Theorem 3.1, from any k-regular Q-borderenergetic graph, we can construct a new class of Q-borderenergetic graphs in the following theorem.

Theorem 6.3 (B. Deng, C. Chang, et al. 2020+)

Let G be a k-regular Q-borderenergetic graph with n vertices. Then $G \nabla \overline{K}_{n-k}$ is Q-borderenergetic.

For a k-regular graph of order n, if G is borderenergetic, then G is Q-borderenergetic and L-borderenergetic. Gong et al.[1] found all the borderenergetic graphs with order $7 \leq n \leq 9$. Bearing in mind that there are no noncomplete borderenergetic graphs with order $n < 7$. Furthermore, Li et al.[2] searched for the borderenergrtic graphs of order 10. Thus, we can find all the regular Q or L-borderenergetic graph of order n, $7 \leq n \leq 10$, see Figure 1. Denote the i-th k-regular Q-borderenergetic graph of order n by $G^i_{n,k}$.

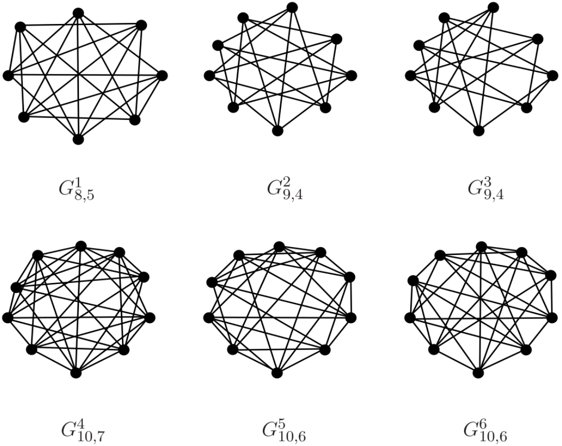


Figure 1. The regular Q-borderenergetic graphs $G^1_{8,5}$, $G^2_{9,4}$, $G^3_{9,4}$, $G^4_{10,7}$, $G^5_{10,6}$ and $G^6_{10,6}$.

Thank you!